Problem Set D, Problem 1

We consider Airy's equation

$$y'' = t\,y$$

with initial conditions $y(0) = 0$, $y'(0) = 1$. We first find the solution.

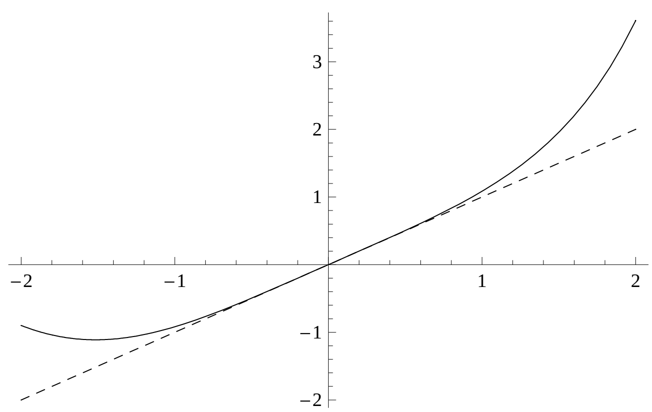
```
y1[t_] = y[t] /.
  First[DSolve[{y''[t] == t*y[t], y[0] == 0, y'[0] == 1}, y[t], t]]
```

$$\frac{1}{6}\left(-3\,3^{1/3}\,\text{AiryAi}[t]\,\text{Gamma}\left[\frac{1}{3}\right] + 3^{5/6}\,\text{AiryBi}[t]\,\text{Gamma}\left[\frac{1}{3}\right]\right)$$

■ (a)

For t close to 0, we want to compare the solution of Airy's equation with the solution to the IVP: $y'' = 0$, $y(0) = 0$, $y'(0) = 1$. The solution of this IVP is $y = t$. Here is a plot of the solution to Airy's equation and the facsimile solution $y = t$. The facsimile solution is plotted as a dashed curve.

```
Plot[{y1[t], t}, {t, -2, 2}, PlotStyle → {Black, {Black, Dashed}}]
```

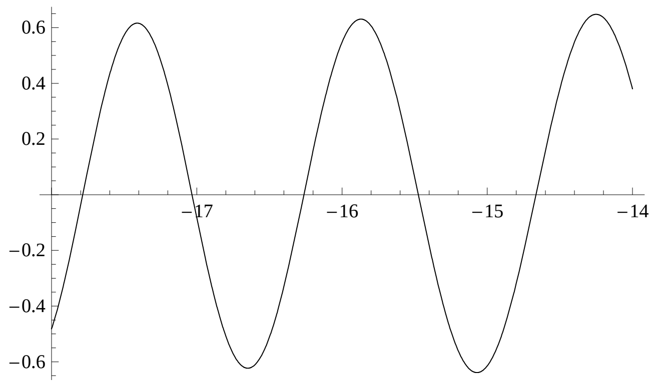


The facsimile solution does indeed agree quite well with the actual solution in a neighborhood of the origin.

■ (b)

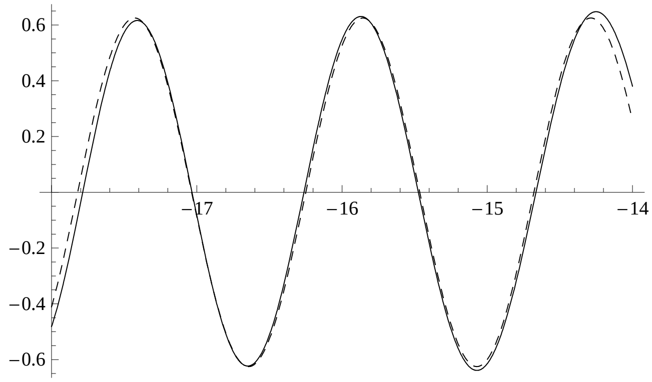
Now for t close to $-16 = -(4^2)$, we want to compare the solution of Airy's equation to a facsimile solution of the form $c_1 \sin(4\,t + c_2)$. We plot the solution of Airy's equation on the interval $(-18, -14)$.

```
Plot[y1[t], {t, -18, -14}, PlotStyle → Black]
```



This certainly looks like a sine wave. Let's see how well it matches up with an appropriate sine wave. We have to choose constants c_1 and c_2 in the facsimile solution to make it match up. Note first that c_1 is the amplitude of the facsimile solution, and we can see from the graph that the amplitude of the solution to Airy's equation is about 0.625 on this interval (see the *Graphics* section of Chapter 8 for instruction on finding coordinates of points in *Mathematica* plots). The constant c_2 determines the phase shift and can be read off from the zeros of the solution. In particular, the solution of Airy's equation has a zero at about -16.25, so we should choose c_2 so that $4 * (-16.25) + c_2 = 0$; i.e. $c_2 = 65$. Here is the plot.

```
Plot[{y1[t], 0.625 Sin[4 t + 65]},
 {t, -18, -14}, PlotStyle → {Black, {Black, Dashed}}]
```

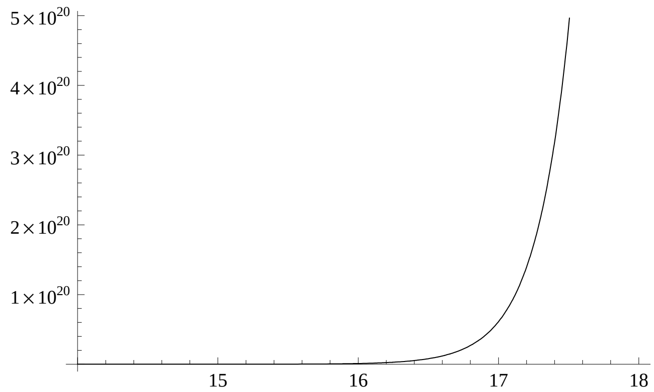


The plots match up very well. Why did we have to choose the values of c_1 and c_2 by hand? Our analysis suggests that the facsimile solution should approximate the actual solution near $t = -K^2$. But the initial condition we've used for Airy's equation is at the point $t = 0$, where the sinusoidal facsimile solutions do not approximate the solution of Airy's equation. Thus the undetermined constants in the facsimile solution do not really have anything to do with the initial conditions in Airy's equation, so we had to choose them by hand to make the solutions match up. Nevertheless, the *frequency* of the facsimile solution is determined by K and is independent of the undetermined constants. Thus we can conclude at least that the frequency of the solutions of Airy's equation in a neighborhood of $t = -K^2$ will be proportional to K.

■ (c)

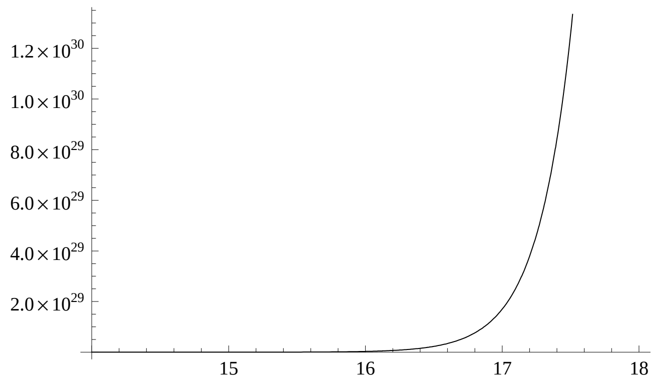
Now for t close to $16 = (4^2)$, we want to compare the solution of Airy's equation to a facsimile solution of the form $c_1 \sinh(4\,t + c_2)$. As in (b), we first plot the numerical solution of Airy'sequation on the interval (14, 18).

```
Plot[y1[t], {t, 14, 18}, PlotStyle → Black]
```



We'd like to compare this graph to that of the hyperbolic sine function. Again, we have to choose c_1 and c_2. Let's make an arbitrary choice this time: $c_1 = 1$, $c_2 = 0$.

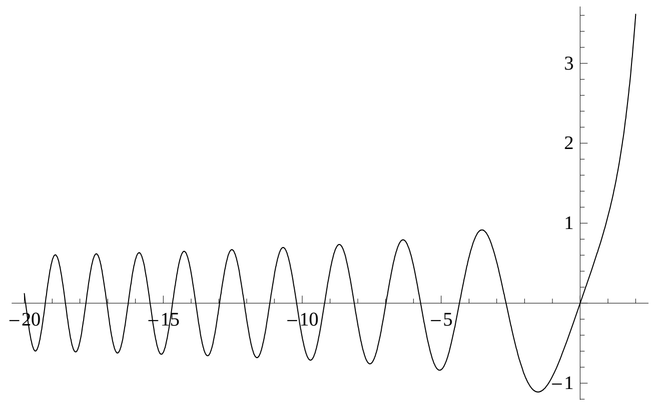
```
Plot[Sinh[4 t], {t, 14, 18}, PlotStyle → Black]
```



The graphs are remarkably similar. Note, however, that the values in the second graph are about $3 * 10^9$ greater than in the first, which means we could choose c_1 to be about $3 * 10^{-10}$.

▪ (d)

```
Plot[y1[t], {t, -20, 2}, PlotStyle → Black]
```



It appears from the graph that the frequency increases and the amplitude decreases as t goes to $-\infty$. The increasing frequency was predicted by our facsimile analysis. The decreasing amplitude is harder to explain.